ANNALS OF MATHEMATICS.

VOL. IV. DECEMBER, 1888. No. 6.

EFFECT OF FRICTION AT CONNECTING-ROD BEARINGS ON THE FORCES TRANSMITTED.

By PROFS. J. BURKITT WEBB* and D. S. JACOBUS, Hoboken, N. J.

Fig. 1 represents the equilibrium of the forces acting on a connecting rod, with a centre line WC and wrist and crank pins indicated by circles about W and C whose radii are respectively r_w and r_c.

Figs. 1′ and 1″ represent the equilibrium of the forces acting at the wrist and crank pins.

The forces shown in these figures are as follows:

$P_a - F_1 =$ pressure of steam less the accelerating force for piston, piston rod, and cross-head;

$G' =$ the pressure of the cross-head guides against the cross-head for frictionless pins, acting at the centre of the wrist pin at the angle $90° + \varphi'$. This angle is the same for frictionless or rough pins;†

$G'_f =$ the pressure when there is friction.

P_c and $P_w =$ pressures of the pins C and W against the rod for frictionless pins;

P_{cf} and $P_{wf} =$ the same when there is friction;

$KG = PH =$ the single force which, if applied in the line NG, will support the weight and correctly accelerate the rod;

I and $J =$ pressures of the ends of the rod at the pin centres W and C due to its acceleration and weight and for frictionless pins;

I' and $J' =$ the same pressures for rough pins acting at the points F and E of the pins, F being the foot of a perpendicular from W on P_{wf} and E the foot of a perpendicular from C on P_{cf}.

The moments of I and I' about N must evidently be equal and opposite to those of J and J' about the same point; from which it follows that

$$I : J = CN : NW$$

and

$$I' : J' = EN'' : N''F.$$

* Professor Jacobus insists upon my name appearing first in this article. I fully appreciate the courtesy, but it is hardly fair to himself, as he has done most of the work. J. B. W.

† $\varphi' =$ angle of friction at cross-head guides.

In addition to the above the following lines will be needed for the purpose of demonstration:

Ee and *Ff* friction circles about C and W, the radii of which are respectively equal to $r_c \sin \varphi$ and $r_w \sin \varphi$;

FE and FC connecting F with E and C;

MD from M, parallel and equal to PK or GH;

DO connecting D with O.

Fig. 2 is composed of various equilibrium polygons representing the equilibrium of the forces acting on various parts, each polygon being drawn with and without friction; these forces are, of course, equal and parallel to those in Fig. 1, *nm* being the force due to the weight and acceleration and therefore the same as KG and PH. Fig. 2′ is an enlargement, for the sake of clearness, of a portion of Fig. 2.

smo is the equilibrium triangle of the forces acting on the wrist pin when there is no friction on wrist or crank pin.

mno is the equilibrium triangle of the forces acting on the rod, as a whole, for frictionless pins, and *mnh* the same when there is friction. The introduction of the force *oq* transmitted by the rod from wrist to crank pin divides *mn* into *mq* and *qn*, forces applied at the wrist and crank pins, thereby cutting *mno* into two polygons, *mqo* representing the equilibrium of the forces acting on the crank pin and *oqn* representing forces on the wrist pin. In the same way *hp* divides the polygon *mnh* into *mph* and *hpn*.

mjo and *njo*, and *mhiq* and *qihn*, are also polygons for wrist and crank pins.*

In addition to the above the following lines will be needed for the purpose of demonstration:

ma perpendicular to P_{wf};

af parallel to CF in Fig. 1;

hf perpendicular to P_{cf};

ne perpendicular to P_{cf};

eh parallel to EF (Fig. 1) through *e*, the intersection of *ne* and *af*, and passing therefore through *h*, as will be shown;

mxz perpendicular to P_{cf} through *m*;

zc parallel to *mn* through *z*, the intersection of *mx* with *ef*;†

* *mq* and *qn* are the simplest set of forces that will support the weight and produce the required acceleration, inasmuch as they cause no unnecessary tension in the rod. Any pair of forces having *mn* for a resultant will give the support and acceleration, consequently we may make various convenient suppositions in regard to these forces, as, for instance, we may suppose that the crank-pin force remains normal to the crank circle, which gives us *mj* and *jn*. The intersection of any such pair of forces will fall on the line $L'S'$, and any pair may be changed to another by introducing two equal and opposite forces at the ends of the rod, thus *qj* combined with *mq* produces *mj* and *jq* combined with *qn* produces *jn*.

† By mistake the figure contains two *c*'s.

nwv perpendicular to P_{wf} through n;

vf' parallel to *mn* through v, the intersection of *nw* with *ef*;

hd perpendicular to P_{wf};

hi perpendicular from h on $L'S'$, which is parallel to the centre line LS, Fig. 1;

bc perpendicular on $L'S'$ from b, the intersection of *hd* with *af*;

hg perpendicular from h on *ef*;

bk perpendicular from b on *hi*;

bkgh a semicircle containing the right angles *bkh* and *bgh*;

kg a line completing the triangle *ghk* which is similar to *dab*, similar to *WCF* (Fig. 1), as will be proved;

nq' through n parallel to the crank radius;

mj connecting m with j the intersection of nq' with *ao*;

or' perpendicular from o on nq';

hq' perpendicular from h on nq'.

Fig. 2 contains the following forces:

$sm = P_a - F_1$;

om and $hm = P_w$ and P_{wf};

on and $hn = P_c$ and P_{cf};

os and $hs = G'$ and G'_f, the guide reactions;

$mn =$ the resultant of the weight of the rod at its centre of gravity and the forces due to the acceleration of the different parts of the rod. *nm* is therefore that single force which, if applied at the proper point, would support and correctly accelerate the rod; *nm* is also the resultant of such parts of the wrist and crank pin pressures as give the support and acceleration;

mp and *mq* and *pn* and $qn =$ the parts of the wrist and crank pin pressures which actually give the support and acceleration;

hp and $oq =$ the force transmitted from wrist to crank pin by the rod, with and without friction, *hp* obliquely from F to E, *oq* axially;

$oh =$ change in guide reaction due to pin friction;

$$bc \text{ and } hk = A \text{ and } B = \frac{P_{wf} r_w \sin \varphi}{nR} \text{ and } \frac{P_{cf} r_c \sin \varphi}{nR} \text{ and}$$

$hi = bc + hk = A + B$, two forces which will be explained in the proper place;

$jo =$ that portion of the force transmitted from the wrist to the crank pin that performs work upon the fly wheel, for frictionless pins;

$qi =$ the force transmitted axially if P_{wf} and P_{cf} be supposed to act at the centres of the pins, and moments be introduced to counterbalance the effect of altering their points of application;

or' = force acting on crank pin in tangential direction for frictionless pins;
hq' = the same when there is friction.

We will suppose the force $P_a - F_1$ applied at the wrist pin W to be known and proceed to determine the force that reaches the crank pin C.

For frictionless pins we proceed as follows:

Construct the polygon *smqo* by making *sm* and *mq* equal and parallel respectively to $P_a - F_1$ and I, and drawing from q and s lines parallel to CW and G', marking their intersection o, then will os be the guide reaction and oq the force transmitted from W to C; om will be P_w the resultant of the forces G' and $P_a - F_1$. Next lay off qn equal and parallel to J and complete the triangle *mno*, then will $no = P_c$, the pressure of the crank pin upon the connecting rod. This pressure P_c is pressure available for doing work upon the crank and fly wheel.

If there is friction the construction of the diagrams is more complicated, for the introduction of friction not only changes the direction and magnitude of the forces, but also alters their points of application.

There are three conditions that govern the construction of a diagram including the effect of friction; these are:

(*a*) The resultant of P_{cf} and P_{wf} must equal in magnitude and direction, and have the same line of action, NG, as the resultant of P_c and P_w;

(*b*) The projections of P_{wf} and P_w on a line perpendicular to the guide reaction must be equal to each other and to that of $P_a - F_1$;

(*c*) The presence of friction changes the forces P_c and P_w into P_{cf} and P_{wf}, which no longer pass through the centres of the pins, but are tangent to circles *Ee* and *Ff* whose radii are constant and respectively equal to $r_c \sin \varphi$ and $r_w \sin \varphi$.

The last of these conditions is generally given by authorities, such as Rankine and Weisbach, and requires therefore no demonstration.*

That the resultant of P_{cf} and P_{wf} should be the same as that of P_c and P_w arises from the fact that either is the single force PH or KG which would support the weight and produce the required acceleration of the rod, if applied at the proper point, and must therefore be independent of the friction of the pins.

The second of these conditions is evident from Fig. 2, where *mos* is the triangle of forces acting on a frictionless wrist pin and *mhs* the triangle with friction considered; obviously P_w and P_{wf} have the same component perpendicular to os as $P_a - F_1$ has.

If we attempt to apply these principles directly to Fig. 1, they lead respectively to the following geometrical conditions:

(*a*) HP must be equal in length and direction to GK and must lie in the same line GN;

(*b*) DO is parallel to G';

*See Rankine's Machinery and Mill Work, p. 428.

(c) PF is perpendicular to $FW = r_w \sin \varphi$, and PE is perpendicular to $EC = r_c \sin \varphi$.

The construction of the diagram under these conditions can only be accomplished by approximate methods more difficult of application than those which we will explain in the construction of Fig. 2.

The geometrical conditions for the construction of Fig. 2 are as follows:

(a) mn is common to both the polygons $mnos$ and $mnhs$;

(b) ohs is a right line.

(c) In applying the third principle we meet with the only difficulty in the construction of Fig. 2; a fundamental principle in mechanics, however, furnishes us with a simple means of solving the problem with all the accuracy desirable.

Inasmuch as the effect of friction is to displace the forces which the pins exert upon the rod, so that they become tangent to the friction circles previously mentioned, these forces will exert moments upon the rod tending to rotate it, and therefore affecting the other forces of the system, so that the forces P_c and P_w become P_{cf} and P_{wf}, besides becoming displaced from C to E and W to F. By a principle in mechanics a force applied at any point, as F, is equivalent to an equal force, in magnitude and direction, at any other point, as W, plus a moment equal to the force multiplied by the perpendicular distance through which it has been displaced. We may, therefore, suppose the forces P_{cf} and P_{wf} to be applied at the centres of the pins if, at the same time, we introduce the moments

$$P_{wf} r_w \sin \varphi \quad \text{and} \quad P_{cf} r_c \sin \varphi.$$

For convenience we will suppose each moment to be produced by a pair of equal and opposite forces acting perpendicular to the rod at W and C. Calling these forces A and B, and letting the distance $WC = nR = n \times$ crank radius, we shall have

$$AnR = P_{wf} r_w \sin \varphi,$$

and

$$BnR = P_{cf} r_c \sin \varphi,$$

so that the moment to be introduced will be

$$AnR + BnR = (A+B)\, nR.$$

According to this, instead of supposing P_{wf} and P_{cf} to act on the rod at F and E, no error will be involved if we suppose them to act at W and C and introduce two additional forces, $A + B$ acting at W perpendicular to WC and $-(A + B)$ acting at C. A and B appear also in the equilibrium polygons Figs. 3 and 4.*

*When the angle β is increasing A is $+$. B is $+$ or $-$ according to whether θ is supposed to increase or diminish.

The important advantage of this change is that it allows the use of the known accelerating forces I and J in place of the unknowns I' and J'.

The construction of Fig. 2 by the aid of $A + B$ is as follows:

Having laid off $P_a - F_1$ and mq as before, and drawn through q and s lines qo and so, respectively parallel to the rod and the guide reaction, cut so by a line parallel to qo and at a distance from it $= A + B$. This will give the point h from which the perpendicular $hi = A + B$ may be let fall upon qo. The polygon $smqih$ will now represent the equilibrium of the forces acting on the wrist pin, and hs will be the new value of the guide reaction. Connecting h with m and n we obtain P_{wf} and P_{cf}.

The value of $A + B$ can be obtained with an exactness amply sufficient by using the values of P_w and P_c in place of those of P_{wf} and P_{cf}, as will be shown in the case of a horizontal high-speed engine, to which formulæ derived by this method have been applied. This form of the result has the advantage that it enables the same to be easily applied.

As the above reasoning may seem to involve a departure from the exact conditions of the problem, in which the forces are really applied at F and E, we will proceed to prove that the introduction of A and B into the solution for frictionless pins will give the correct result when friction is considered, that is;

To prove that $hi = A + B$.

This proof divides itself into three parts as follows:

(a) to prove that $cb = A$;

(b) to prove that $kh = B$;

(c) to prove that $ih = kh + cb$.

(a) From the similar triangles CFW and abd we have

$$\frac{WF}{CW} = \frac{db}{ad};$$

but

$$ad = \frac{P_{wf}}{\cos \mu},$$

because am and dk are perpendicular to mh, and ad makes with mh the angle μ, also

$$CW = nR,$$

and

$$WF = r_w \sin \varphi,$$

substituting these values, we have

$$db = \frac{P_{wf} r_w \sin \varphi}{nR \cos \mu};$$

but

$$cb = db \cos \mu;$$

$$\therefore cb = \frac{P_{wf} r_w \sin \varphi}{nR} = A.$$

(b) To prove that $kh = B$.

To prove this we must first show the following:

I. That a line through e parallel to EF divides mn at p into the two forces $mp = I'$, and $pn = J'$ acting at F and E.

II. That the line ep produced will pass through h.

III. That hgk is similar to CWF.

I. To show that mn is correctly divided at p, that is to show that

$$mp = I', \text{ and } pn = J'.$$

The relations previously given between I, I', J, and J' are

$$I + J = I' + J' = mn,$$

$$I : J = CN : NW,$$

and

$$I' : J' = EN'' : N''F.$$

By construction, therefore, $mn = I' + J'$, and we have only to prove that

$$mp : pn = I' : J'.$$

First, to prove that

$$mr : nr = CN : NF':$$

q being the point of division for frictionless pins, we know that

$$mq : qn = CN : NW,$$

and also, from the similar triangles amq and wnq, we have

$$mq : qn = aq : qw;$$

consequently

$$aq : qw = CN : NW.$$

This, in addition to the similarity of the triangles CFW with avw, CFF' with avf', and $CN'N$ with arq, gives us two similar figures contained in Figs. 1 and 2; viz:

$$CN'FF'WNC,$$

similar to

$$arvf'wqa,$$

from which it follows that

$$CN : NF' = aq : qf'.$$

But

$$aq : qf' = ar : rv,$$

because arq and avf' are similar, and by the similar triangles mra and nrv we have

$$ar : rv = mr : nr;$$

$$\therefore CN : NF' = mr : nr,$$

which was to be proved.

Second, to prove that

$$mp : pn = EN'' : N''F:$$

From the similar triangles *ner* and *mzr* we have

$$mr : rn = zr : re;$$

but

$$mr : rn = CN : NF'$$

$$= CN' : N'F;$$

$$\therefore zr : re = CN' : N'F.$$

This, in addition to the similarity of the triangles $FE'C$ with *ecz*, *FEC* with *exz*, and CEE' with *zxc*, gives us two similar figures contained in Figs. 1 and 2; viz:

$$FN'CE'EN''F$$

similar to

$$erzcxpe,$$

from which it follows that

$$xp : pe = EN'' : N''F.$$

From the similar triangles *mxp* and *nep* we have

$$mp : pn = xp : pe;$$

$$\therefore mp : pn = EN'' : N''F;$$

but

$$EN'' : N''F = I' : J';$$

$$\therefore mp : pn = I' : J',$$

which was to be proved.

II. To show that *ep* passes through *h*.

At F the force P_{wf} of the pin against the rod is divided into I' and the force transmitted from F to E, or, in other words, P_{wf} is the resultant of the latter forces, and when reversed will form with them a triangle of forces. This triangle *mhp* appears in Fig. 2, in which $pm = I'$ and $mh = - P_{wf}$, and the third side, having been drawn through p and parallel to FE, must pass through h, so as to make *hp* the force transmitted from F to E.

III. To show that *hgk* is similar to *CWF*.

The angle *kgh* is equal to $hgb + bgk = 90° + \mu$, because *hgb* is inscribed in the semicircle *bgh* and $bgk = bhk = \mu$; also, *khg* is equal to $fao = \psi$, because *gh* and *kh* are perpendicular to *fa* and *oa*; therefore the triangle *hkg* has two of its angles the same as the angles of *CWF*, in which $FCW = \psi$ and $CWF = 90° + \mu$, it being the exterior angle of the right angle triangle *WFS*.

We are now prepared to prove that $kh = B$. From the similar triangles FEC and ehf we have

$$\frac{hf}{ef} = \frac{CE}{CF};$$

but
$$CE = r_c \sin \varphi,$$

and
$$ef = \frac{P_{cf}}{\cos \alpha};$$

$$\therefore hf = \frac{P_{cf} r_c \sin \varphi}{CF \cos \alpha};$$

but
$$gh = hf \cos \alpha;$$

$$\therefore gh = \frac{P_{cf} r_c \sin \varphi}{CF}.$$

From the similar triangles hgk and CWF we have

$$\frac{gh}{kh} = \frac{CW}{CF} = \frac{nR}{CF};$$

$$\therefore kh = gh \frac{CF}{nR}.$$

Substituting the value of gh and reducing, we have

$$kh = \frac{P_{cf} r_c \sin \varphi}{nR} = B,$$

as was to be proved.

(c) To prove that $ih = kh + cb$.

This follows because bk is perpendicular to hi, and therefore $bc = ki$; consequently, as

$$cb = A \quad \text{and} \quad kh = B,$$

$$ih = A + B,$$

as was to be proved.

ANALYTICAL EXPRESSION OF THE FOREGOING PRINCIPLES.

Let $R =$ radius of crank circle;

$nR =$ length of connecting rod;

$lR =$ distance from the wrist pin to the foot of the perpendicular let fall from the centre of gravity of the rod upon its centre line;

$cR =$ distance of the centre of gravity from the centre line of the rod;

θ and $\beta =$ angles of crank and connecting rod with the path of the wrist pin;

$\delta =$ angle made by tipping the engine up about the crank shaft;

$\tan \varphi =$ coefficient of friction at wrist and crank pins;

$\tan \varphi' =$ coefficient of friction at cross-head guides;

$\tau =$ angular velocity of crank;

W_1 and $M =$ weight and mass of the piston, piston rod, and cross-head;

W_2 and $m =$ weight and mass of the connecting rod;

$C =$ component of the weight of the piston, piston rod, and cross-head that acts in the direction of the centre line of the cylinder;

D and $E =$ portions of the weight of the connecting rod borne respectively by the wrist and crank pins;

$H =$ friction of the piston and piston rod;

X_1 and $X_2 =$ components, in the direction of the line of travel of the wrist pin, of the accelerating forces at the wrist and crank pins;

Y_1 and $Y_2 =$ components of the accelerating forces at the wrist and crank pins at right angles to X_1 and X_2;

$P_a =$ force produced by the pressure of the steam on the piston;

$F_1 =$ force required to produce the acceleration of the mass of the piston, piston rod, and cross-head;

G and $G_f =$ normal component of the reaction of the cross-head guides for frictionless and rough pins;

N_f and $T_f =$ components of P_{cf} parallel and perpendicular to the crank;

$T_f' =$ the latter reduced to the centre of the crank pin.

We now have $$C = W_1 \sin \delta.$$

The parts of the weight of the connecting rod supported at wrist and crank pins are

$$D = \frac{n - l + c \tan(\beta - \delta)}{n} W_2,$$

and

$$E = \frac{l - c \tan(\beta - \delta)}{n} W_2.$$

Combining these forces with P_a, the accelerating forces, and those due to friction, we obtain the equilibrium polygons, represented in Figs. 3 and 4, for the forces acting on the wrist and crank pins.

From these polygons we obtain the following values of the several forces:*

$$G_f = \frac{1}{1 + \tan \varphi' \tan \beta} \left\{ \left[\begin{matrix} P_a + C - F_1 - H + D \sin \delta \\ -(A + B) \sin \beta - X_1 \end{matrix} \right] \tan \beta \atop + Y_1 + D \cos \delta - (A + B) \cos \beta \right\}$$

$$P_{wf} = \sqrt{[(P_a + C - F_1 - H - G_f \tan \varphi')^2 + G_f^2]},$$

* The method of obtaining these forces, together with those required to produce the acceleration, will be given in detail in an article yet to be published.

$$T_f = \left\{ \begin{array}{l} P_a + C - F_1 - H - G_f \tan \varphi' \\ + D \sin \delta - (A + B) \sin \beta - X_1 \end{array} \right\} \sec \beta \sin (\theta + \beta)$$

$$- Y_2 \cos \theta - X_2 \sin \theta - (A + B) \cos (\theta + \beta) - E \cos (\theta + \delta),$$

$$N_f = \left\{ \begin{array}{l} P_a + C - F_1 - H - G_f \tan \varphi' \\ + D \sin \delta - (A + B) \sin \beta - X_1 \end{array} \right\} \sec \beta \cos (\theta + \beta)$$

$$+ Y_2 \sin \theta - X_2 \cos \theta + (A + B) \sin (\theta + \beta) + E \sin (\theta + \delta),$$

$$P_{cf} = \sqrt{(T_f^2 + N_f^2)},$$

$$T'_f = T_f - Bn.$$

APPLICATION OF THE FOREGOING FORMULÆ.

We will apply these formulæ to the case of a horizontal high-speed engine, assuming an excessive coefficient of friction at the wrist and crank pins, and show that the values of A and B are determined by one approximation to as great a degree of accuracy as is desirable.

The dimensions of the engine are

Length of stroke, 12″;

Diameter of cylinder, 10″;

Length of connecting rod, 36″;

Distance from the wrist pin to the centre of gravity of the rod, 20″.15;

Principal radius of gyration, 15″.00;

Diameter of crank pin, 3″;

Diameter of wrist pin, $2\frac{1}{2}$″;

Weight of piston, piston rod, and cross-head, 90 lbs.;

Weight of connecting rod, 70 lbs.;

Sin φ, .24.

Neglecting friction the forces contained in Table I are obtained.

TABLE I.

Forces in pounds per square inch of piston area.

θ	P_a	F_1	Y_1	Y_2	X_1	X_2	G	T	N	P_w	P_c
0	+ 83	+ 20.50	.0	0	+ 6.85	+ 7.82	0	0	+ 47.8	+ 62.5	+ 47.8
30	+ 81	+ 16.69	− .50	− 3.33	+ 5.63	+ 6.71	+ 4.43	+ 33.1	+ 41.0	+ 64.4	+ 52.6
50	+ 76	+ 10.81	− .76	− 5.10	+ 3.72	+ 4.89	+ 7.17	+ 51.7	+ 26.4	+ 65.6	+ 58.1
70	+ 54	+ 3.74	− .93	− 6.26	+ 1.41	+ 2.49	+ 6.84	+ 48.4	+ 2.7	+ 50.7	+ 48.4
90	+ 33	− 2.97	− .99	− 6.66	− .85	− .17	+ 5.20	+ 37.0	− 12.9	+ 36.4	+ 39.2
110	+ 20	− 8.28	− .93	− 6.26	− 2.70	− 2.74	+ 4.00	+ 27.9	− 22.1	+ 28.6	+ 35.5
130	+ 1	− 11.79	− .76	− 5.10	− 4.00	− 4.95	+ 1.41	+ 12.0	− 19.5	+ 12.9	+ 22.9
150	− 25	− 13.74	− .50	− 3.33	− 4.78	− 6.54	− 1.04	− 2.4	− 1.4	− 11.3	− 2.7
180	− 80	− 14.64	0	0	− 5.17	− 7.48	0	0	+ 52.7	− 65.4	− 52.7

As a first approximation; i. e. letting

$$A_1 = \frac{P_w r_w \sin \varphi}{nR} \quad \text{and} \quad B_1 = \frac{P_c r_c \sin \varphi}{nR},$$

we obtain the values of A and B given in Table II.

TABLE II.

Forces in pounds per square inch of piston area.

θ	A_1	B_1	$A_1 + B_1$	θ	A_1	B_1	$A_1 + B_1$
0	+ .521	+ .478	+ .999	110	− .238	+ .355	+ .117
30	+ .537	+ .526	+ 1.063	130	− .108	+ .229	+ .121
50	+ .547	+ .581	+ 1.128	150	− .094	+ .027	− .067
70	+ .422	+ .484	+ .906	180	− .545	+ .527	− .018
90	0	+ .392	+ .392				

Making use of the values of A and B given in Table II, we obtain the forces when friction is included as given in Table III.

TABLE III.*

Forces in pounds per square inch of piston area.

θ	G_f	$G - G_f$	N_f	$N - N_f$	T_f'	$T - T_f'$	P_{cf}	$P_c - P_{cf}$	P_{wf}	$P_w - P_{wf}$
0	− 1.0	+ 1.00	+ 47.8	0	− 3.9	+ 3.88	+ 47.8	− .01	+ 62.5	+ .01
30	+ 3.4	+ 1.06	+ 41.5	− .53	+ 29.0	+ 4.08	+ 52.5	+ .13	+ 64.4	+ .06
50	+ 6.0	+ 1.13	+ 27.3	− .87	+ 47.6	+ 4.22	+ 57.9	+ .18	+ 65.5	+ .11
70	+ 5.9	+ .92	+ 3.6	− .86	+ 45.2	+ 3.21	+ 48.2	+ .24	+ 50.6	+ .11
90	+ 4.8	+ .40	− 12.5	− .40	+ 34.6	+ 2.35	+ 39.1	+ .13	+ 36.4	+ .05
110	+ 3.8	+ .12	− 22.0	− .11	+ 25.8	+ 2.06	+ 35.5	+ .07	+ 28.6	+ .05
130	+ 1.3	+ .11	− 19.4	− .09	+ 10.7	+ 1.28	+ 22.9	+ .04	+ 12.9	+ .01
150	− .9	− .09	− 1.4	+ .03	− 2.5	+ .21	− 2.7	.00	− 11.3	.00
180	.0	+ .02	+ 52.7	0	− 3.0	+ 3.18	− 52.7	.00	− 65.4	.00

*The computation of this table is facilitated by employing the following equations:

$G - G_f = (A + B) \sec \beta,$

$N - N_f = (A + B) \tan \beta \cos (\theta + \beta) - (A + B) \sin (\theta + \beta),$

$T - T_f' = (A + B) \tan \beta \sin (\theta + \beta) + (A + B) \cos (\theta + \beta) + Bn,$

$$P_w - P_{wf} = \frac{P_w^2 - P_{wf}^2}{P_w + P_{wf}},$$

$$P_c - P_{cf} = \frac{P_c^2 - P_{cf}^2}{P_c + P_{cf}}.$$

Let the values of A and B as determined by a second approximation be A_2 and B_2; then will

$$A_1 - A_2 = \frac{(P_w - P_{wf})r_w \sin \varphi}{nR},$$

and

$$B_1 - B_2 = \frac{(P_c - P_{cf})r_c \sin \varphi}{nR}.$$

Table IV contains values of $A_1 - A_2$ and $B_1 - B_2$.

TABLE IV.

Forces in pounds per square inch of piston area.

θ	$A_1 - A_2$	$B_1 - B_2$	θ	$A_1 - A_2$	$B_1 - B_2$	θ	$A_1 - A_2$	$B_1 - B_2$
0	.0001	.0001	70	.0009	.0024	130	.0001	.0004
30	.0005	.0013	90	.0004	.0013	150	.0000	.0000
50	.0009	.0018	110	.0004	.0007	180	.0000	.0000

The greatest difference between the values of A and B, as obtained by a first and second approximation, is for $\theta = 70°$, at which position $(A_1 + B_1) - (A_2 + B_2) = .0033$ lbs. per square inch of piston area, or $.0036 (A + B)$.

The effect of such an error in $A + B$ upon the result may be found as follows:

The introduction of the forces $(A + B) = .905$ for $\theta = 70°$ alters the value of P_c by the amount of .24 lbs. per square inch of piston area; a variation of $A + B$ equal to $.0036 (A + B)$ will, therefore, cause this difference in P_c to vary about .0036 times itself, or .0009 lbs. per square inch of piston area. This is equivalent to a variation in P_{cf} of .000019 P_{cf}; i. e. *the value of P_{cf} as obtained by one and two approximations differs by about .00002 times itself.* The variation in P_{wf} for $\theta = 70°$, obtained by a similar method of reasoning, is about .000007 times itself. It is evident, therefore, that the value of $A + B$ as determined by the first approximation is sufficiently accurate.